Exercise 43

Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

$$f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ e^x & \text{if } 0 \le x \le 1 \\ 2-x & \text{if } x > 1 \end{cases}$$

Solution

$x+2$ is a polynomial, e^x is an exponential function, and $2-x$ is a polynomial. All of these are continuous on their respective domains by Theorem 7. Any points of discontinuity, then, can only occur at the endpoints of the intervals that these functions are defined on. Check $x = 0$ first.

$$\lim_{x \to 0^-} f(x) \stackrel{?}{=} \lim_{x \to 0^+} f(x) \stackrel{?}{=} f(0)$$

$$\lim_{x \to 0^-} (x+2) \stackrel{?}{=} \lim_{x \to 0^+} e^x \stackrel{?}{=} e^x\Big|_{x=0}$$

$$\lim_{x \to 0^-} x + \lim_{x \to 0^-} 2 \stackrel{?}{=} e^{\lim_{x \to 0^+} x} \stackrel{?}{=} e^0$$

$$0 + 2 \stackrel{?}{=} e^0 = e^0$$

$$2 \neq 1 = 1$$

The condition for $f(x)$ to be continuous at $x = 0$ is not satisfied. Therefore, $f(x)$ is discontinuous at $x = 0$ but is continuous from the right. Check $x = 1$ next.

$$f(1) \stackrel{?}{=} \lim_{x \to 1^-} f(x) \stackrel{?}{=} \lim_{x \to 1^+} f(x)$$

$$e^x\Big|_{x=1} \stackrel{?}{=} \lim_{x \to 1^-} e^x \stackrel{?}{=} \lim_{x \to 1^+} (2-x)$$

$$e^1 \stackrel{?}{=} e^{\lim_{x \to 1^-} x} \stackrel{?}{=} \lim_{x \to 1^+} 2 - \lim_{x \to 1^+} x$$

$$e^1 = e^1 \stackrel{?}{=} 2 - 1$$

$$e = e \neq 1$$

The condition for $f(x)$ to be continuous at $x = 1$ is not satisfied. Therefore, $f(x)$ is discontinuous at $x = 1$ but is continuous from the left.

Below is a graph of $f(x)$ versus x.